Visualising and the Move from Informal to Formal Linear Measurement

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Forty-three children, aged 8 and 9, were interviewed about their understanding of informal and formal measurement. Results showed that they visualised units accurately if the number of units was small. For many this was their preferred method of measuring. When asked to iterate an informal unit or use rulers in unconventional ways, they made errors that indicated ignorance of many of the unwritten rules of measurement. Successful children used visualization and the width of a finger to measure.

Linear measurement entails comparison, either of two quantities or by using intermediary non-standard or standard units for this comparison. As several authors have summarised, this requires appreciating the need for conservation of length, an understanding of units and when they can be subdivided for measuring continuous quantity, why and how units can be iterated, and the appropriate use of measuring tools (e.g., Nunes & Bryant, 1996; Outhred & McPhail, 2000; Piaget, Inhelder & Szeminska, 1960).

Age-related differences have been shown in understanding of measurement. Nunes and associates carried out a series of studies with children aged 5 and 6 years many of whom are quite naïve about measurement (summarised in Nunes & Bryant, 1996). Bragg & Outhred (2000) interviewed children from Grades 1 to 5 and showed interesting changes in this age range, especially between Grades 2 and 3. The study reported here was concerned with students at the age, in New Zealand, when they usually move from informal to formal measurement (*Mathematics in the New Zealand Curriculum*, Ministry of Education, 1992).

Children may appear to make the transition to formal linear measurement easily, as they are already familiar with the most common measuring tool, the ruler. This familiarity may disguise the need to understand the principles given above. To measure correctly they must understand that the length of units is conserved in that all units are the same length, whether informal or formal. They need to understand that when they put one unit down after another in iteration, the two units must just touch and not overlap. Gaps are not allowed. Some of these concepts are built into rulers but need to be observed when the object to be measured is longer than the ruler. Formal measuring instruments add other important and difficult concepts, namely that in the use of standard measuring instruments measurement starts at 0 and scales can be used to give fractional parts of continuous length.

One reason why formal measurement is difficult may lie in the limited opportunities that our children have outside school in which precision in measurement is required. Out of school, children may help measure ingredients for baking or may measure how tall they are (an excellent task for comparison) but they rarely engage in activities that require precision. In the Philippines, people use a phrase "puwede na yan" which can be translated as "that's good enough". This concept of "good enough" is not limited to the Philippines, but is common in most areas of measurement and especially those that children have experience with. In asking the 43 students in this study where they used measurement outside of

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 358-365). Sydney: MERGA.

school, only one mentioned measurement in which precision was needed. This child spoke of working with his father, who was always building things around the house. That child might be expected to know how to use a measuring tape and the importance of starting with 0. The other 42 children either could think of no examples of when they measured outside of school or gave examples in which precision was not necessary. As adults do less tailoring or carpentry at home, children see fewer models out of school that require precision.

Method

The authors interviewed 43 children, from four schools, individually using five tasks and standard initial questions but with follow-up questions chosen to cover predetermined topics. Twenty-two children were between age 8 years and 8 years 11 months (called age 8) and 21 were between 9 years 0 months and 9 years 9 months (called age 9). Interviews lasted 15 to 20 minutes. Careful notes were kept on students' activities during the interview on recording sheets that prompted interviewers to explore and record children's answers, their use of visualization, their measuring processes, their use of units and fractional units, and their experience with measuring. For items involving a ruler, specific notes were made on appreciation of the need for 0 at the start of a ruler or in measuring, where the child started measuring from, and uniformity of units where theses were not given. Children also wrote on their sheets. Teachers were also interviewed.

The first two items required children to visually judge the number of ribbons or tiles would fit in a given length and then use informal methods to measure. A third item required them to finish drawing a ruler than someone else had started drawing. A fourth item asked them to measure four dimensions with a "broken ruler". Items 3 and 4 were adapted from Nunes and Bryant (1996). The final task asked them to give the length of string with an s-shaped bend in it that was pictured next to a ruler (task from the TIMMS-R).

Students came from four schools of different character, as we expected different results from children who attended these schools. One was a small Montessori school, one was an expensive private girls' school, one was a small lower middle class primary school, and one was a large primary school from an upper middle class area. We found no major differences in the accuracy of children between schools, which in itself was surprising. Therefore data from all schools are given together. All teachers followed a national curriculum for measurement, which for these ages covers the transfer from informal measurement to formal measurement. Where there were small differences that might be the result of teaching these are referred to in the results and discussion.

Results and Discussion

Children's responses were analysed in relation to concepts needed for measurement, especially the unwritten rules for units and iteration.

Analysis of Responses to Tasks 1 and 2, Use of Informal Measurement.

The first item required students to estimate how many ribbons could be cut from a given length when both were pictured on their sheets. Then were then asked to check their estimate but given no tools for this.

Table 1

Number of Children, Age 8 and Age 9, giving the Listed Responses when asked How Many Ribbons could be cut from a Strip

Response	Age 8		Age 9	
	For visual estimate	By informal measuring	For visual estimate	By informal measuring
3	8	2	17	5
3 with a bit leftover	4	17	3	15
4	8	1	1	1
Other responses	2 (5,1½)	2 (18, 1½)	0	0

Note: The strip given was 3.4 times the length of the ribbon.

A higher percentage of 9-year-old children (19 or 95%) than 8-year-old children (12 or 55%) were accurate for this visual estimate. It could be argued that the children who estimated 4 rather than 3 were not far off. This shows very good accuracy in visualization of length in this age group. Asked how they did this, one girl said *"I looked at it from that bit and then I tried to get the same amount"*. The unusual responses like 18 and 1 ½ were the result of children losing sight of the interviewers' goal for this task.

Most children measured with their fingers or with their pencil. Some did not measure with an instrument but continued to trust their visualization. Thus given a measuring task with no tools, and asked both to visualise and then to measure, these young children showed marked accuracy, in a comparative measurement of this size.

On the next task they were again asked to estimate visually and then were given two tiles and asked to measure the length, including parts of tiles that would be needed to just fill the length.

Table 2

Number of Children, Age 8 and Age 9, giving the Listed Responses when asked to Visualise and Measure How Many Tiles Would be Needed to Fill a Space and Leave no Gaps.

Response	Age 8		Age 9	
	Visual estimate	Measure	Visual estimate	Measure
4 (and parts)	10	3	8	1
5 (and parts)	8	18 (all with parts)	9	15 (13 with parts)
6	3	1	2	5
Other	1 (7)		2 (7, 3½)	

Note: The strip was 5.11 times the length of the tiles

This was a more difficult measure to visualise, possibly because more units fit into the space. About the same percentage of 8 and 9-year-old children were accurate in their estimate, 36% and 43% respectively. It was interesting that 45% of the 8-year-olds and 38% of the 9-year-olds underestimated, but few children overestimated when asked to visualise.

Measuring was to be done with two tiles that could be alternated to measure the space. Children's manual dexterity may have influenced their accuracy. Despite this, 82% of the 8-year-old children and 71% of the 9-year-old children measured reasonably accuracy.

However, even reasonably accurate answers demonstrated some misconceptions. Errors in measurement using the two given tiles were related to failing to respect the unspoken rules of iteration. They included:

- The need for each unit to be right next to the previous one. This unspoken rule was broken by children who overlapped the tiles so that six tiles fit in without any fractional bits left over. It was also broken by a child who laid his pencil down to mark the end of one tile, and laid the next tile on the other side of the pencil.
- The need to count only completely iterated units, and give the remaining portion a fractional name. Placing a final tile over the boundary so that a space slightly greater than 5 tiles was called 6 broke this rule.
- The need to count all iterations. For example, one child counted 4 iterations, counting only those put down after the first tile.
- The need to count each unit or partial unit only once. This rule was broken by children who laid a final tile over the right-hand edge, and counted it both as six and as ¼. For example, one child said that the answer could be 5 ¼ or 6 ¼. She measured 5 ¼ but unsure whether or not to count the partial one as both 6 and ¼.

Other methods were not wrong but failed to take advantage of having a measuring unit. Of children who did this, one continued to visualise with considerable accuracy, deciding that 5 ½ tiles would be needed to fit the space. Another used both tiles to iterate as one unit, pushing them both along rather than alternating. One measured with her fingers spread to the width of a tile. One put both tiles down, but then used only the second one to iterate.

There were both careless measurers who had to be reminded to be careful, and some very careful children. For example, one boy who had initially estimated 4¾ measured once and got 5, measured a second time and got 4 7/8 and measured a third time, getting 5 1/8. For an actual answer of 5.11 or 5 1/9 this was very accurate indeed.

In an attempt to get at children's understanding of fractional parts of units, we asked what they would call the leftover bit if there were one. The children either gave the leftover portion an informal name, like a bit or a slice, or used a common fraction. A quarter, a half and a third were all used sensibly. Others called the bit either a cm or – surprisingly in a country that abandoned imperial measurement 35 years ago - an inch. Sometimes these terms were used together, suggesting that the child had an idea that there were words for bits but was uncertain what they were. For example, when asked what was left over, one girl said *"½ - a cm? 2 cm?"*

Analysis of Children's Responses to Item 3, Finishing a Drawing of a Ruler

The third task explored children's understanding of the most common formal or standard linear measuring instrument in their environment, a ruler. This task explored whether or not they could show an understanding that units on a ruler needed to be of the same length and whether or not they understood the importance of starting with 0 for a complete initial unit. The drawing that they were given had lines for 3, 4, 5, and 6 with marks halfway between these. Only the numbers 4, 5, and 6 were given. If completed correctly this ruler would have had 0 on the left margin, without a space before it, if their units were of the same size.

The rulers used in these New Zealand schools are identical. They all have centimetres on one side and millimetres on the other starting from the other end. Either "cm" or "mm" is printed at the left of the ruler and there is no 0. The marks do not start at the left end of the ruler but start about 5 mm in. When children completed the drawing of a ruler on this task they did not have their school rulers present, but they were very likely to have a mental image of this tool as they use it daily, for underlining if not for measuring. For measuring they must learn which of the aspects of their rulers are essential to measurement and which can be ignored. With these factors, some errors in their use are not surprising. For example, in a previous study (Irwin, 2001) one child, when asked what came between 0 and 1, said that "cm" came between them and an older child said that there was no relation between centimetres and millimetres because they were on opposite sides of the rule and went in opposite directions.

Drawings were analysed for evidence that they appreciated that the first unit ended with 1, and that all units were similar is size.

Table 3
Starting Point Marked on Rulers Drawn by Children Age 8 and Age 9.

Starting point of drawn ruler	Age 8	Age 9
Acceptable Drawings		
0	8	7
cm where 0 should have been	1	2
Unacceptable Drawings		
Notable space before 0 or cm	3	7
No mark before 1	3	2
Starting at 1	1	2
0 placed at the 0.5 mark	5	0
Nothing before 2	0	1
Units very uneven	1	0

Thus less than half of the students in either age group started their rulers in a way that showed understanding that the first unit was the same size as other units and started with 0 (41% and 43% respectively).

A separate analysis was made of the evenness of the units that children drew. This was done in analysing only the portions before 3 and not those at the right hand end. Some leeway was allowed if the child's intent was clear. Half of the children in each age group drew adequately regular units (50% of Age 8 and 52% of Age 9), suggesting that these children had an understanding of this essential aspect of measurement units. Those who made units of irregular size usually did so because of confusions about where the numbers should start.

Children who were unsuccessful on this task ignored unwritten rules of measurement that

- All units must be the same size.
- "1" marks the end of the first unit.

Teaching appears to have made a difference for some of these children. Seven of the 15 students who started their rulers with 0 at the very start of the space were from one class. There were also several children from the same class who made a variety of incorrect drawings (starting at 1, putting the 0 at the 0.5 position, or putting no marks before 1), so if teaching did aid these seven students, it did not prevent other errors.

Responses to Task 4, Measuring Four Lengths with a Broken Ruler

For this task children were given a portion of a ruler that ran from 3 to 7.5 cm on one edge and from 236 to 287 mm on the other edge. This portion of a ruler was made from a photocopy of their school rulers. This item was included to allow us to observe children's understanding of units when given a portion of a ruler that did not start at or near 0. They were asked to measure the height and width of a pictured house and then the height and width of its door. The first two dimensions were longer than the ruler and the second two were shorter than the ruler.

We wanted to see if they counted the spaces for units rather than the lines or numbers, and what technique they used for iterating the broken ruler when it was not long enough. Photocopying warp made the measurements more difficult than intended, but not beyond the capabilities of careful children. The correct answers to measuring four dimensions were 5.6 cm, 8 cm, 3.3 cm and 2.8 cm. One child of 8 measured these with a broken ruler and got the answers: 5 cm and 7 mm, 7 cm and 9 mm, 3 cm and 2 mm and 2 cm and 8 mm.

Table 4
Students Measuring each Segment with Enough Accuracy to Show that they were Using the Units on the Broken Ruler.

Response	Item a [5.6]		Item b [8]		Item c [3.3]		Item d [2.8]	
	Age 8	Age 9	Age 8	Age 9	Age 8	Age 9	Age 8	Age 9
Accurate	12	9	9	7	7	7	8	10
1 cm up – counting lines	6	7	6	6	13	10	7	7
Inaccurate	4	5	7	8	2	4	7	4

Note: Accuracy for Item a [5.6] was considered 5 to 6, For Item b [8] 7.5-8.5, for Item c [3.3] 2.8 – 3.8, and for Item d [2.8] 2.3 – 3.3.

The differences between age groups were not marked in most cases. More 8-year-olds than 9-year-olds were accurate on Item a. The majority of these children did not iterate the ruler but imagined the units that would come before the 3 on the ruler, an effective strategy. However, visualisation was not always used successfully. Some students reported visualising all the lines before the start of the broken bit and counting them, ignoring the unit that came before the first line.

Visualising on Item b was more difficult as both the beginning and end of the ruler would have to be pictured. Children were more likely to iterate the ruler, which caused more confusion in counting lines or spaces. These two techniques were the source of some of the quite inaccurate measurements.

On the third and fourth measurements there was no need to either visualise spaces or to iterate. Many more students referred only to the numbers on the ruler without reflecting on what was missing. Some students made the same measuring technique on these two items, and others changed their method of measuring, suggesting that they were not certain of what to do. We were reminded by their activity that this was a school task, without the payoffs for incorrect measurement that occur when precision is needed in everyday life.

Some of the answers coded as 'quite inaccurate' were the result of reading from the mm side of the ruler, without understanding the units or iterating. For example one boy gave the fourth reading as "263 mm". Others were reasonably accurate in their measurement but did not know what to call the units. For example "*7 and 9 inches – no millimetres – I forget the names*".

Two successful children used the width of a finger to measure one cm when this was needed before or after the broken ruler. When this successful method was described to their teacher, the teacher reported that they had spent time measuring various body parts, and the width of a child's finger at this age was usually 1 cm. This seemed an excellent way of transferring a skill gained in informal measurement to formal measurement.

Responses To Task 5, Measuring a Picture of a Folded String Placed Next to a Ruler.

This task required students to use the units of a pictured ruler to measure a string with an s-shaped bend in it that was drawn above the ruler. The total to be measured was 7 cm, made up of portions of 3 cm, 1 cm, and 3 cm. Again, students were required to know that the space of the unit, not the lines, were what was to be measured.

Table 5
Methods Used by Children, Age 8 and Age 9, in Measuring Segments of a Picture of a Folded String above a Picture of a Ruler.

Technique used	Age 8		Age 9	
	Technique	Accurate	Technique	Accurate
Visualising	10	1	8	1
Measuring	11	1	12	3
Visualising and measuring	1	0	1	0

Some children who visualised on this task reported that they stretched out the string in their mind, sometime to the beginning of the ruler and sometimes to 1. For example *"I brought the end* [mentally] *to cm and thought it would probably be that long"*

Very few children measured accurately. As shown above, only two of the 8-year-old and four of the 9-year-old children were accurate in their measurement. Most children could see that the middle piece was 1 cm, looking at the space between the lines. For the longer segments they counted lines or stretched back visually only to 1. Children who measured matched a spread of their fingers to the ruler, used their finger to measure 1 cm, or lined up the segments with the ruler in the picture.

Implications

Children demonstrated a hierarchy of skills on these increasingly difficult tasks. Their first preference appears to have been for visualising. We were surprised how frequently and how accurately these young children visualised to get linear measurements. Even when not asked to estimate they often reverted to picturing units when measurement with iterated tiles or a ruler was difficult. When the number of units to be visualised were small they did well, both on estimating that three ribbons could be cut from a longer strip and on the task with the broken ruler where they had to estimate the length of 3½ cm. However, while visualising worked for picturing units it was not effective for picturing how long a string would be if pulled straight, a much longer distance.

Next to this preference for visualising came using their fingers to measure, either a finger's width, which was very close to 1 cm, or fingers spread to match a given length. The first of these measurements was very accurate, but the latter often did not conserve the distance when moved to a different place.

Iterating informal units to measure was moderately successful, but in doing this several students showed that they were uncertain of the unwritten rules of measuring units. Some of the same unwritten rules were broken in drawing a ruler as in measuring with two tiles. Errors in drawing and using a ruler may be exacerbated by the rulers that the children use daily. However, confusions about whether the spaces or the lines were to be counted could not be blamed on the rulers that they used.

Accurate understanding of the use of rulers in the unusual situations in these interviews proved the most difficult skill to use, especially in the last task in which neither the ruler nor the object to be measured could be moved. These children were starting to use rulers for standard measurement, but many were not transferring the rules for units and iteration from informal measurement.

Visualizing called upon their experience of what would fit in a space. This may be a task that children have much more experience with than they do with precise measurement, as Owens and Outhred (1998) found for measurement of area. It depends on a spatial skill similar to that used in putting puzzles together. As this group reported almost no situations in which they had to use precise measurement outside school, this must be seen as a school task rarely reinforced out of school.

When the children's teachers were interviewed, they were not surprised by the children's weaknesses. Some thought that formal measurement was introduced too early, reflecting their understanding of these children's difficulties. However, they were pleased to hear of the successful use of a finger width as an intermediary between informal and formal measuring. Similarly, the teacher who had emphasised the role of 0 on a ruler had taught a useful piece of knowledge.

These interviews suggest that children's skills in visualisation should be fostered. They also suggest that the use of body lengths can help a successful transfer from informal to formal measurement.

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